

# AN UPDATE ON CARTESIAN PRODUCTS WITH FEW DISTINCT DISTANCES

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**ABSTRACT.** Every set of points  $\mathcal{P}$  determines  $\Omega(|\mathcal{P}|/\log |\mathcal{P}|)$  distances. A close version of this was initially conjectured by Erdős in 1946 and rather recently proved by Guth and Katz. We show that when near this lower bound, a point set  $\mathcal{P}$  of the form  $A \times A$  must satisfy  $|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|$ . This improves recent results of Hanson and Roche-Newton. We also prove an asymmetric version for cartesian products of the form  $A \times B$ . Both are meant to provide further evidence that sets with few distinct distances exhibit some additive structure.

## 1. INTRODUCTION

Let  $\mathcal{P}$  be a set of points in plane, and let  $\Delta(\mathcal{P})$  denote the set of squares of distances spanned by  $\mathcal{P}$ . In other words,

$$\Delta(\mathcal{P}) = \{(p_1 - q_1)^2 + (p_2 - q_2)^2 : (p_1, p_2), (q_1, q_2) \in \mathcal{P}\}.$$

In [5], Guth and Katz showed that  $\Delta(\mathcal{P}) \gg |\mathcal{P}|/\log |\mathcal{P}|$ , where  $\gg$  represents the usual Vinogradov symbol. When  $P = A \times B$  for some finite sets of reals  $A$  and  $B$ ,  $\Delta(A \times B) = (A - B)^2 + (A - B)^2$ , so this says that

$$|(A - B)^2 + (A - B)^2| \gg \frac{|A||B|}{\log |A||B|}.$$

In [3], Erdős originally conjectured that all sets  $\mathcal{P}$  should determine  $\Omega(|\mathcal{P}|/\sqrt{\log |\mathcal{P}|})$  distinct distances, so the Guth-Katz bound is almost optimal. Nonetheless, very little is known for sets that achieve this bound. It is widely believed that sets with  $O(|\mathcal{P}|/\log |\mathcal{P}|)$  distinct distances should come from some type of lattice. This is very well-motivated by the following beautiful result of Bernays [2], which generalizes a classical theorem of Landau.

**Theorem 1.** *Let  $f(x, y) = ax^2 + bxy + cy^2$  for integers  $a, b, c \in \mathbb{Z}$ , such that the determinant  $b^2 - 4ac$  is not an integer square. Then, the number of integers between 1 and  $n$  that can be expressed as  $f(u, v)$  with  $u, v \in \mathbb{Z}$  is  $O(n/\sqrt{\log n})$ .*

Using Theorem 1, one can easily check that sets with  $O(n/\sqrt{\log n})$  distinct distances are given by  $\sqrt{n} \times \sqrt{n}$  subsets of the integer lattice, the (equilateral) triangular lattice, or, more exotically, by the rectangular lattice

$$\mathcal{L}_r = \{(i, j\sqrt{r}) \mid i, j \in \mathbb{Z}, 1 \leq i, j \leq n\},$$

for every integer  $r > 1$ . We refer the reader to [10] for a more detailed presentation of this discussion, where Sheffer also points out that unlike the first two examples, the lattices  $\mathcal{L}_r$  do not span squares or equilateral triangles.

In this paper, we will only take a look at sets that come from cartesian products, and show that whenever they determine few distinct distances they must exhibit some additive structure. Specifically, when  $\mathcal{P} = A \times A$ , we show that when the Guth-Katz bound is close to being tight, we have that

$$|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|.$$

Unfortunately, the argument we use to show this does not extend to the asymmetric case. When  $\mathcal{P} = A \times B$  (and the Guth-Katz bound is close to being tight), we were however able to use some different ideas to show that

$$\min \{|A - A|, |B - B|, |A - B|\} \ll (|A||B|)^{1-\frac{13}{205}} \cdot L(A, B),$$

where

$$L(A, B) = \min \left\{ \log^{\frac{3}{205}} |A|, \log^{\frac{3}{205}} |B| \right\}.$$

Even though this is weaker than the inequality we obtained for the symmetric case, it is in some satisfying since most of the extremal configurations that Theorem 1 suggests come from asymmetric cartesian products.

We state both of these results more formally below.

**Theorem 2.** *Suppose  $A$  is a finite set of real numbers and let  $\Delta(A \times A)$  be the set of distances spanned by  $A \times A$ . Then,*

$$|A - A| \ll |\Delta(A \times A)| |A|^{-\frac{2}{7}} \log^{\frac{1}{7}} |A|.$$

The bound obtained improves a recent theorem by Hanson [6], who showed that

$$|A - A| \ll |\Delta(A \times A)| |A|^{-\frac{1}{8}}.$$

In the meantime this was also sharpened by Oliver Roche-Newton in [9], who showed

$$|A - A| \ll |\Delta(A \times A)| |A|^{-\frac{2}{11}},$$

but Theorem 1 is tighter. Our proof will rely on the very simple observation that for every four real numbers  $a_1, a_2, b_1, b_2$ , we have that

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 - (b_1 - a_2)^2 - (b_2 - a_1)^2 = 2(a_2 - a_1)(b_1 - b_2),$$

which yields the inclusion

$$2 \cdot D \cdot D \subset 2D^2 - 2D^2.$$

We will combine this with the beautiful sum-product estimate of Solymosi [12]:

**Theorem 3.** *Let  $S \subset \mathbb{R}$  be a set. Then,*

$$|S + S|^2 |SS| \geq \frac{|S|^4}{4 \lceil \log |S| \rceil}.$$

For the asymmetric case, we will prove the following

**Theorem 4.** *Suppose  $A$  and  $B$  are finite sets of real numbers and let  $\Delta(A \times B)$  be the set of distances spanned by  $A \times B$ . Then,*

$$\min \{|A - A|, |B - B|, |A - B|\} \ll (\Delta(A \times B))^{1-\frac{13}{205}} \cdot L(A, B),$$

where

$$L(A, B) = \min \left\{ \log^{\frac{3}{205}} |A|, \log^{\frac{3}{205}} |B| \right\}.$$

The argument will rely on two results. The first one is the following beautiful Lemma by Balog [1], which comes from Solymosi's original idea from Theorem 2.

**Lemma 5.** *Let  $R, S, T$  be finite sets of real numbers. Then*

$$|RT + RT||ST + ST| \gg |R/S||T|^2.$$

The second one is by Shkredov and is Theorem 3 in [11].

**Lemma 6.** *Let  $A \subset \mathbb{R}$  be a finite set and let  $D = A - A$ . Then*

$$|D/D| \gg |D|^{1+\frac{1}{12}} \log^{-1/4} |D|.$$

We are now ready for the proofs.

## 2. PROOF OF THEOREM 2

For convenience, suppose that  $|D^2 + D^2| = |\Delta(A \times A)| \ll |A|^2$ . Consequently, it is enough to prove that

$$|D| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|.$$

We apply Theorem 2 for the set  $S := D^2 = \{(x - y)^2 : x, y \in A\}$ . Using the observation that  $|D^2 D^2| = |DD|$ , this yields

$$\begin{aligned} |D^2 + D^2|^2 |DD| &= |D^2 + D^2|^2 |D^2 D^2| \\ &\geq \frac{|D^2|^4}{4 \lceil \log |D^2| \rceil} \\ &= \frac{|D|^4}{4 \lceil \log |D| \rceil}. \end{aligned}$$

On the other hand

$$2 \cdot DD \subset 2D^2 - 2D^2,$$

so Plünnecke's inequality gives

$$\begin{aligned} |D^2 + D^2|^2 |DD| &= |D^2 + D^2|^2 |2 \cdot DD| \\ &\leq |D^2 + D^2|^2 |2D^2 - 2D^2| \\ &\ll |D^2 + D^2|^2 \left( \frac{|D^2 + D^2|^4}{|D|^3} \right) \\ &\ll \frac{|A|^{12}}{|D|^3}. \end{aligned}$$

Putting the two bounds together, we conclude that

$$\frac{|A|^{12}}{|D|^3} \gg \frac{|D|^4}{4 \lceil \log |D| \rceil},$$

which yields

$$|D| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|.$$

□

### 3. PROOF OF THEOREM 4

For convenience, suppose again that  $|(A-B)^2 + (A-B)^2| = |\Delta(A \times B)| \ll |AB|$ . Since

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 - (b_1 - a_2)^2 - (b_2 - a_1)^2 = 2(a_2 - a_1)(b_1 - b_2)$$

holds for every  $a_1, a_2 \in A$ ,  $b_1, b_2 \in B$ , we have the inclusion

$$2 \cdot (A - A) \cdot (B - B) + 2 \cdot (A - A) \cdot (B - B) \subset 4(A - B)^2 - 4(A - B)^2.$$

On one hand Plünnecke's inequality gives

$$\begin{aligned} |4(A - B)^2 - 4(A - B)^2| &\leq \frac{|(A - B)^2 + (A - B)^2|^8}{|(A - B)^2|^7} \\ &\ll \frac{|A|^8 |B|^8}{|(A - B)^2|^7} \\ &= \frac{|A|^8 |B|^8}{|A - B|^7}. \end{aligned}$$

On the other hand, the above inclusion gives

$$\begin{aligned} |4(A - B)^2 - 4(A - B)^2|^2 &\geq |2 \cdot (A - A) \cdot (B - B) + 2 \cdot (A - A) \cdot (B - B)|^2 \\ &= |(A - A) \cdot (B - B) + (A - A) \cdot (B - B)|. \end{aligned}$$

Furthermore, Lemma 5 applied for  $R = S = A - A$ ,  $T = B - B$  tells us that

$$|(A - A) \cdot (B - B) + (A - A) \cdot (B - B)|^2 \gg \left| \frac{A - A}{A - A} \right| |B - B|^2.$$

By Lemma 6,

$$\left| \frac{A - A}{A - A} \right| \gg |A - A|^{1+\frac{1}{12}} \log^{-1/4} |A - A|,$$

so

$$|4(A - B)^2 - 4(A - B)^2|^2 \gg |A - A|^{1+\frac{1}{12}} |B - B|^2 \log^{-1/4} |A - A|.$$

We conclude that

$$\frac{|A|^{16} |B|^{16}}{|A - B|^{14}} \gg |A - A|^{1+\frac{1}{12}} |B - B|^2 \log^{-1/4} |A - A|.$$

By using Lemma 4 for  $R = S = B - B$  and  $T = A - A$  instead, we can similarly get

$$\frac{|A|^{16} |B|^{16}}{|A - B|^{14}} \gg |B - B|^{1+\frac{1}{12}} |A - A|^2 \log^{-1/4} |B - B|.$$

Putting everything together, we get

$$\min\{|A - A|, |B - B|, |A - B|\} \ll (|A||B|)^{1-\frac{13}{205}} \cdot \mathsf{L}(A, B),$$

where

$$\mathsf{L}(A, B) = \min \left\{ \log^{\frac{3}{205}} |A|, \log^{\frac{3}{205}} |B| \right\}.$$

This completes the proof. □

## 4. SOME REMARKS

Theorem 2 is still far from being optimal. We conjecture that when  $|\Delta(A \times A)| \ll |A|^2$ ,

$$|A - A| \ll |A|^{1+\epsilon},$$

for any  $\epsilon > 0$ .

It is worth mentioning that even assuming the full-strength of the Erdős-Szemerédi conjecture [4], which says that for any  $\epsilon > 0$  one has

$$\max |D^2 + D^2|, |D^2 D^2| \gg |D|^{2-\epsilon},$$

the argument above only gives

$$|A - A| \ll |A|^{2-\frac{4}{7}+\epsilon}.$$

Using the updates of Konyagin and Shkredov on Solymosi's bound [7,8], one can perhaps bring

$$|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|$$

down to

$$|A - A| \ll |A|^{2-\frac{2}{7}-c} \log^{\frac{1}{7}} |A|$$

for some small constant  $c > 0$ , but significant improvements to Theorem 2 should not come from improvements to Theorem 3.

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